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Quasi-static and dynamic response of viscoelastic helical rods

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Abstract

In this study, the dynamic behaviour of cylindrical helical rods made of linear viscoelastic materials are investigated in the Laplace domain. The governing equations for naturally twisted and curved spatial rods obtained using the Timoshenko beam theory are rewritten for cylindrical helical rods. The curvature of the rod axis, effect of rotary inertia, and shear and axial deformations are considered in the formulation. The material of the rod is assumed to be homogeneous, isotropic and linear viscoelastic. In the viscoelastic material case, according to the correspondence principle, the material constants are replaced with their complex counterparts in the Laplace domain. Ordinary differential equations in scalar form obtained in the Laplace domain are solved numerically using the complementary functions method to calculate the dynamic stiffness matrix of the problem. In the solutions, the Kelvin model is employed. The solutions obtained are transformed to the real space using the Durbin's numerical inverse Laplace transform method. Numerical results for quasi-static and dynamic response of viscoelastic models are presented in the form of graphics.

1. Introduction

The dynamic behaviour of helical bars and curved rods is an important engineering problem. In practice, helical bars are used as structural elements known as helical stairs and as mechanical elements in vehicle suspension systems and motor valve springs. To simplify the analysis, it is generally assumed that the material is elastic. However, in reality, the materials are vicoelastic due to internal friction, and thus the viscoelastic constitutive relations yield more realistic results than the elastic constitutive relations with regard to the material behaviour.

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In many research papers, the dynamic response of viscoelastic materials are investigated using various models.

The application of the Laplace transform to viscoelastic beams was presented by Flügge [1]. Kiral et al. [2] presented the equations of motion for viscoelastic curved rods, although they did not solve the problem effectively.

Findley et al. [3] used the correspondence principle and the superposition principle for solving the governing equations of the viscoelastic beam. Christensen [4] reported the transient response of the viscoelastic beam using the Fourier transform. The above studies are based on the fact that the governing equations of viscoelasticity can be converted to the equations of elasticity by integral transformations. For complex geometries and constitutive relations, closed-form solutions are often not possible and numerical solution techniques should be adopted.

The application of the finite element method to the complex geometry has been presented by a number of authors. White [5] used the constitutive law of hereditary integral type, in which the time interval form is approximated by the finite difference method to perform a finite element analysis in a quasi-static problem. Adey and Brebbia [6] used an approximate inversion procedure to obtain the inversion solution of the associated elastic problem. Chen and Lin [7] studied the dynamic response of a beam using a creep law of time hardening to model the viscoelastic material. Yamada et al. [8] reported the natural frequency of a viscoelastic beam and a rod.

Chen [9] studied the linear viscoelastic Timoshenko beam for quasi-static and dynamic response. He assumed that the Poisson ratio is constant and only the elasticity modulus is viscoelastic. The relaxation modulus is expressed by the same Prony series for both normal stress–strain and shear stress–strain relations. The hybrid method is used to remove the time parameter using the Laplace transform and the associated equation is solved using the finite element method.

Aköz and Kadioğlu [10] examined a mixed finite element for elastic circular beams using Gâteaux differential. Using a similar approach, Aköz and Kadioğlu [11] studied the quasi-static and dynamic analysis of viscoelastic Timoshenko and Euler–Bernoulli beams. Kadioğlu and Aköz [12] also studied the general forms of relaxation modulus for both the Poisson ratio and Young modulus for quasi-static and dynamic response of circular beams. In order to remove the time derivatives from the governing equations and boundary conditions, the method of the Laplace–Carson transform was utilized.

Ilyasov and Aköz [13] examined static and dynamic behaviour of plates. The viscoelastic constitutive equations were written in the Boltzmann–Volterra form.

Park and Schapery [14] presented and tested a numerical method of interconversion between modulus and compliance functions when the given and predicted functions are based on a Prony series representation of transient functions. Schapery and Park [15] proposed and verified a simple approximate interconversion method by examples. Park [16] examined different approaches to the mathematical modelling of viscoelastic dampers and compared their theoretical basis and performance.

Kim and Kim [17] studied the parametric instability of a laminated beam subjected to a periodic loading. The governing equations were derived from Hamilton's principle with Bolzmann's superposition principle for linear viscoelastic constitutive equations.

Furthermore, the viscoelastic behaviour of materials has been studied by researchers. Various methods using three-parameter solid, Maxwell and Kelvin models have been presented for the analysis of such problems [18–26].

As mentioned above, the viscoelastic models are commonly used in structures like straight beams, plates and shells. However, to the best of the present authors' knowledge the viscoelastic analysis of helical bars has not been reported yet.

In this study, quasi-static and dynamic response of viscoelastic helical rods under time-dependent loads are investigated in the Laplace domain. The governing equations for naturally twisted and curved spatial rods obtained using the Timoshenko beam theory and Frenet's formulation are rewritten for cylindrical helical rods as in Refs. [27,28]. The curvature of the rod axis, effect of rotary inertia, and shear and axial deformations are considered in the formulation. More advanced beam theories include the effects of initial twist and curvature (see, e.g. Refs. [29–33]). The aim of the present work is to demonstrate the application of an efficient method to the viscoelastic case rather than developing an advanced beam theory including the initial twist and curvature effects on the kinematic and constitutive relations. The formulation to be developed will include the initial twist and curvature effects on the kinematics but exclude their effects on the constitutive equations as in Refs. [27,28]. In the viscoelastic material case, according to the correspondence principle, the material constants are replaced with their complex counterparts in the Laplace domain. Viscoelastic counterparts of both the Poisson ratio and elasticity modulus are used in the formulation. The dynamic stiffness matrix of the problem is calculated in the Laplace transform space by applying the complementary functions method [34] to the differential equations in canonical form. The solutions obtained in the Laplace domain are then transformed to the time space using the Durbin's inverse Laplace transform method [35–37]. This provides great convenience in the solution of the problems having general boundary conditions. The desired accuracy is obtained by taking only a few elements as opposed to high number of elements (in the order of hundreds) needed in finite element analysis. Ordinary differential equations with variable coefficients can also be solved exactly in the Laplace domain by using the complementary functions method. In the solution of viscoelastic helical rods, the Kelvin model is considered. Numerical results for elastic-static, quasi-static, elastic-dynamic and viscoelastic dynamic responses of helical rods are presented.

2. Rod geometry

Consider a naturally curved and twisted spatial slender rod. The trajectory of geometric centre G of the rod is defined as the rod axis and its position vector at $t = 0$ is given by $\mathbf{r}^o = \mathbf{r}^o(s, 0)$, where s is measured from an arbitrary reference point $s = 0$ on the axis (Fig. 1a).

Let, at any time t, a moving reference frame be defined by unit vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ with the origin of the axis of the rod is chosen such that

$$\mathbf{t} = \frac{\partial \mathbf{r}^o(s,t)}{\partial s}, \tag{1}$$

where $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are unit tangent, normal and binormal vectors, respectively. The following differential relations among the unit vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ can be obtained with the aid of the Frenet formulae [38]

$$\partial \mathbf{t}/\partial s = \chi \mathbf{n}, \quad \partial \mathbf{n}/\partial s = \tau \mathbf{b} - \chi \mathbf{t}, \quad \partial \mathbf{b}/\partial s = -\tau \mathbf{n}, \tag{2}$$

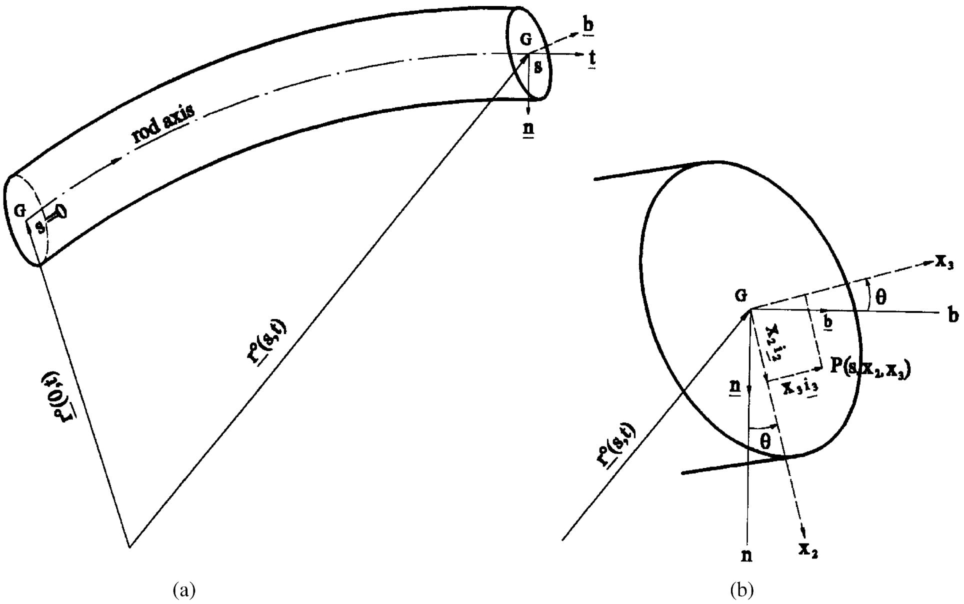

Fig. 1. The rod geometry.

where χ and τ are the curvature and the natural twist of the axis, respectively. A second rectangular frame (x_1, x_2, x_3) is introduced such that the x_1 axis is in the direction of $\mathbf{t}$, and x_2, x_3 axes are the principal axes of the cross-section (Fig. 1b). Let $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ be the unit vectors along x_1, x_2, x_3. From Fig. 1b Eq. (3) can be written as

$$\mathbf{t} = \mathbf{i}_1, \quad \mathbf{n} = \mathbf{i}_2 \cos\theta - \mathbf{i}_3 \sin\theta, \quad \mathbf{b} = \mathbf{i}_2 \sin\theta + \mathbf{i}_3 \cos\theta. \tag{3}$$

3. Governing equations

Let the displacement of a point on the rod axis, and the rotation of the cross-section about an axis passing through G be denoted by $\mathbf{U}^o(s,t)$ and $\mathbf{\Omega}^o(s,t)$, respectively. Also, let $\gamma^o(s,t)$ and $\boldsymbol{\omega}^o(s,t)$ denote extension and rotation of the unit length on the rod axis, respectively.

On the other hand, let $\mathbf{T}(s,t)$ and $\mathbf{M}(s,t)$ denote, respectively, the resultant of the internal stresses acting on the cross-section, and the resultant moment obtained when $\mathbf{T}(s,t)$ is carried to the geometric centre G. Also let $\mathbf{p}^{ex}(s,t)$ and $\mathbf{m}^{ex}(s,t)$ be the external distributed load and moment per unit length of the rod axis.

Assuming infinitesimal deformations, the equations of geometric compatibility and the equations of motion are, respectively, given by [39]

$$\gamma^o = \frac{\partial \mathbf{U}^o}{\partial s} + \mathbf{t} \times \mathbf{\Omega}^o, \quad \boldsymbol{\omega}^o = \frac{\partial \mathbf{\Omega}^o}{\partial s}, \tag{4}$$

$$\frac{\partial \mathbf{T}^o}{\partial s} + \mathbf{p}^{(ex)} = \mathbf{p}^{(in)}, \quad \frac{\partial \mathbf{M}^o}{\partial s} + \mathbf{t} \times \mathbf{T}^o + \mathbf{m}^{(ex)} = \mathbf{m}^{(in)}. \tag{5}$$

The inertia force and moment $\mathbf{p}^{in}$ and $\mathbf{m}^{in}$ are defined as

$$p_i^{(in)} = -\rho A \frac{\partial^2 U_i^o}{\partial t^2}, \quad m_i^{(in)} = -\rho I_i \frac{\partial^2 \Omega_i^o}{\partial t^2} \quad (i = t, n, b), \tag{6}$$

where ρ is the mass density.

Assuming that the centroid and the shear centre of cross-section coincide; that the normal and binormal axes are the principal axes; that the effect of warping is ignored; and that the material of the rod is homogeneous, linear elastic and isotropic, then the constitutive equations are

$$T_i^o = A_{ij}\gamma_j^o, \quad M_i^o = D_{ij}\omega_j^o \quad (i,j = t, n, b). \tag{7}$$

A_{ij} and D_{ij} are defined as

$$[A] = \begin{bmatrix} EA & 0 & 0 \\ 0 & GA/\alpha_n & 0 \\ 0 & 0 & GA/\alpha_b \end{bmatrix}, \tag{8}$$

$$[D] = \begin{bmatrix} GI_t & 0 & 0 \\ 0 & EI_n & 0 \\ 0 & 0 & EI_b \end{bmatrix}, \tag{9}$$

where A is the cross-sectional area, E and G are the elastic constants, α_n and α_b are the shear correction factors, I_t is the torsional and I_n, I_b are the bending moments of inertia. For α_n and α_b the Reissner's correction factor 6/5 is used in the present analysis. Note that although the effects of initial twist and curvature are included in the kinematic equations, these effects are neglected in the constitutive equations [27,28].

Equations of geometric compatibility and equations of motion are derived under the assumption that the displacements and their gradients are infinitesimal. The largest dimension of the cross-section is small compared to the radii of curvature and twist of the rod axis. Also, the effect of warping of the cross-section is ignored.

The equations of geometric compatibility (4) and the equations of motion (5) are valid irrespective of the constitution of the rod material. Thus, there are four vectorial equations in six vectorial unknowns, namely, $\mathbf{U}^o$, $\mathbf{\Omega}^o$, $\mathbf{T}^o$, $\mathbf{M}^o$, $\boldsymbol{\gamma}^o$ and $\boldsymbol{\omega}^o$. The remaining two equations necessary for the determination of these unknowns are the constitutive equations.

4. Laplace transforms of the governing equations

For the case of forced vibrations, a column matrix $\mathbf{Y}(s,t)$ is introduced as

$$\mathbf{Y}(s,t) = \{U_t^o,\ U_n^o,\ U_b^o,\ \Omega_t^o,\ \Omega_n^o,\ \Omega_b^o,\ T_t^o,\ T_n^o,\ T_b^o,\ M_t^o,\ M_n^o,\ M_b^o\}^{\mathrm{T}}. \tag{10}$$

The Laplace transform of Eq. (10) with respect to time $L[\mathbf{Y}(s,t)] = \bar{\mathbf{Y}}(s,z)$, for $t > 0$ is defined as

$$\bar{\mathbf{Y}}(s,z) = \int_0^\infty \mathbf{Y}(s,t)\mathrm{e}^{-zt}\,\mathrm{d}t, \tag{11}$$

where the Laplace transform parameter z is a complex number. With the aid of these definitions, Eqs. (4) and (5) are reduced to a set of 12 first order non-homogeneous ordinary differential equations

$$\frac{\mathrm{d}\bar{\mathbf{Y}}(s,z)}{\mathrm{d}s} = \bar{\mathbf{F}}(s,z)\bar{\mathbf{Y}}(s,z) + \bar{\mathbf{B}}(s,z). \tag{12}$$

Some of the elements of $\bar{\mathbf{F}}(s,z)$ are obtained by applying the Laplace transform of the following second derivatives

$$\begin{aligned} L\left[\rho A \frac{\partial^2 U_k^o}{\partial t^2}\right] &= \rho A\left[z^2 \bar{U}_k^o - zU_k^o(s,0) - \frac{\partial U_k^o(s,0)}{\partial t}\right] \\ L\left[\rho I_k \frac{\partial^2 \Omega_k^o}{\partial t^2}\right] &= \rho I_k\left[z^2 \bar{\Omega}_k^o - z\Omega_k^o(s,0) - \frac{\partial \Omega_k^o(s,0)}{\partial t}\right] \end{aligned} \quad (k = t, n, b). \tag{13}$$

The second and third terms on the right side of Eq. (13) are the initial conditions given at $t = 0$. The elements of the column matrix $\bar{\mathbf{B}}(s,z)$ are

$$\begin{aligned} \bar{B}_i(s,z) &= 0, \quad (i = 1, 2, \ldots, 6), \\ \bar{B}_{6+j}(s,z) &= -(\bar{p}_k^{(ex)}) - \rho A\left[zU_k^o(s,0) + \frac{\partial U_k^o(s,0)}{\partial t}\right] \quad (j = 1, 2, 3), \\ \bar{B}_{9+j}(s,z) &= -(\bar{m}_k^{(ex)}) - \rho I_k\left[z\Omega_k^o(s,0) + \frac{\partial \Omega_k^o(s,0)}{\partial t}\right] \quad (k = t, n, b). \end{aligned} \tag{14}$$

Note that the initial conditions present in Eqs. (13) are now included in the load vector $\bar{\mathbf{B}}(s,z)$.

5. Effect of damping

The case of internal viscoelastic damping is treated with the help of the correspondence principle, as described, for example, in Ref. [40]. The constitutive law of the Kelvin viscoelastic model is

$$s_{ij} = 2G\left(e_{ij} + g\frac{\mathrm{d}e_{ij}}{\mathrm{d}t}\right), \tag{15}$$

where G is the shear modulus and g is the internal damping coefficient. Here, the deviatoric stress and strain tensors s_{ij} and e_{ij} are defined in terms of the stress and strain tensors σ_{ij} and ε_{ij}, respectively, by

$$s_{ij} = \sigma_{ij} - \delta_{ij}\tfrac{1}{3}\sigma_{kk}, \quad e_{ij} = \varepsilon_{ij} - \delta_{ij}\tfrac{1}{3}\varepsilon_{kk}, \tag{16}$$

with δ_{ij} being the Kronecker's delta and repeated indices indicating summation. The correspondence principle can be stated as follows: the Laplace transform of the viscoelastic solution can be obtained from the Laplace transform of the elastic solution by replacing the elastic constants G and λ (Lamè constant) by [40]

$$G_v = G(1 + gz), \quad \lambda_v = \lambda + \frac{2}{3}(G - G_v), \tag{17}$$

or the elastic constants E and v by

$$E_v = \frac{E(1 + gz)}{1 + [(1 - 2v)/3]gz}, \quad v_v = \frac{3v - (1 - 2v)gz}{3 + (1 - 2v)gz}, \tag{18}$$

where E_v and v_v are viscoelastic material constants, and z is the Laplace transform parameter.

6. Special cases

The spatially curved system is taken as a special case of a helical bar. The parametric equation of a helix is [34] (see Fig. 2)

$$x = a\cos\phi, \quad y = a\sin\phi, \quad z = h\phi, \tag{19}$$

where ϕ is the horizontal angle of the helix. The infinitesimal length element of the helix is defined as

$$c = \sqrt{a^2 + h^2}, \quad \mathrm{d}s = c\mathrm{d}\phi, \quad \cos\alpha = \frac{a}{c}, \quad \sin\alpha = \frac{h}{c}, \tag{20}$$

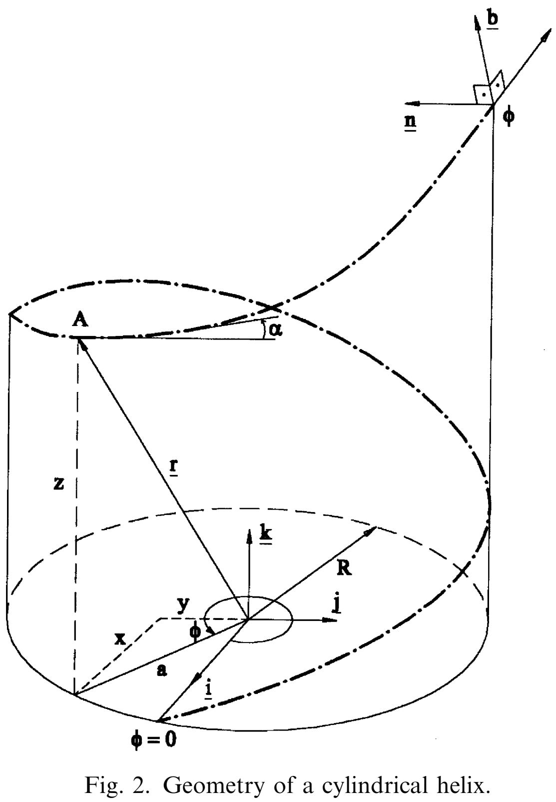

Fig. 2. Geometry of a cylindrical helix.

where α and a are the pitch angle and centreline radius of the helix, respectively. The curvatures of a cylindrical helical spring are

$$\chi = \frac{a}{c^2} = \text{constant}, \quad \tau = \frac{h}{c^2} = \text{constant}. \tag{21}$$

The relationship between the moving triad $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ and the fixed reference frame $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are (Fig. 2)

$$\{\mathbf{V}\}^{\mathrm{T}}_{tnb} = [\mathbf{B}]\{\mathbf{V}\}^{\mathrm{T}}_{ijk} \left\{ \begin{array}{c} V_t \\ V_n \\ V_b \end{array} \right\} = \begin{bmatrix} -(a/c)\sin\phi & (a/c)\cos\phi & (h/c) \\ -\cos\phi & -\sin\phi & 0 \\ (h/c)\sin\phi & -(h/c)\cos\phi & (a/c) \end{bmatrix} \left\{ \begin{array}{c} V_i \\ V_j \\ V_k \end{array} \right\}, \tag{22}$$

where V_i, V_j, V_k are the vector components in the fixed reference frame and V_t, V_n, V_b are the vector components with respect to the moving triad.

Non-dimensional parameters in the Laplace domain are defined as

$$\bar{U}_i = \frac{1}{c} U^o_i, \quad \bar{\Omega}_i = \Omega^o_i, \quad \bar{T}_i = \frac{c^2}{EI_n} T^o_i, \quad \bar{M}_i = \frac{c}{EI_n} M^o_i \quad (i = t, n, b). \tag{23}$$

It is assumed that the $\mathbf{n}, \mathbf{b}$ axes become the principal axes, and the effect of warping of the cross-section is ignored. Now, equations obtained as a result of elimination of $\boldsymbol{\gamma}^o$ and $\boldsymbol{\omega}^o$ between the transformed equations of compatibility (4) and the transformed constitutive equations (7) together with the transformed equations of motion (5) form the governing equations of the dynamic response of initially curved and twisted viscoelastic bars. Finally, using Eqs. (20), (21) and (23), the governing equations in canonical form are given as follows

$$\frac{\mathrm{d}\bar{U}_t}{\mathrm{d}\phi} = \frac{a}{c}\bar{U}_n + \frac{EI_n}{E_v A c^2}\bar{T}_t, \tag{24a}$$

$$\frac{\mathrm{d}\bar{U}_n}{\mathrm{d}\phi} = -\frac{a}{c}\bar{U}_t + \frac{h}{c}\bar{U}_b + \bar{\Omega}_b + \frac{\alpha_n EI_n}{G_v A c^2}\bar{T}_n, \tag{24b}$$

$$\frac{\mathrm{d}\bar{U}_b}{\mathrm{d}\phi} = -\frac{h}{c}\bar{U}_n - \bar{\Omega}_n + \frac{\alpha_b EI_n}{G_v A c^2}\bar{T}_b, \tag{24c}$$

$$\frac{\mathrm{d}\bar{\Omega}_t}{\mathrm{d}\phi} = \frac{a}{c}\bar{\Omega}_n + \frac{EI_n}{G_v I_t}\bar{M}_t, \tag{24d}$$

$$\frac{\mathrm{d}\bar{\Omega}_n}{\mathrm{d}\phi} = -\frac{a}{c}\bar{\Omega}_t + \frac{h}{c}\bar{\Omega}_b + \frac{E}{E_v}\bar{M}_n, \tag{24e}$$

$$\frac{\mathrm{d}\bar{\Omega}_b}{\mathrm{d}\phi} = -\frac{h}{c}\bar{\Omega}_n + \frac{EI_n}{E_v I_b}\bar{M}_b, \tag{24f}$$

$$\frac{\mathrm{d}\bar{T}_t}{\mathrm{d}\phi} = \frac{\rho A c^4 z^2}{EI_n}\bar{U}_t + \frac{a}{c}\bar{T}_n + \bar{B}_7, \tag{24g}$$

$$\frac{\mathrm{d}\bar{T}_n}{\mathrm{d}\phi} = \frac{\rho A c^4 z^2}{EI_n}\bar{U}_n - \frac{a}{c}\bar{T}_t + \frac{h}{c}\bar{T}_b + \bar{B}_8, \tag{24h}$$

$$\frac{\mathrm{d}\bar{T}_b}{\mathrm{d}\phi} = \frac{\rho A c^4 z^2}{EI_n}\bar{U}_b - \frac{h}{c}\bar{T}_n + \bar{B}_9, \tag{24i}$$

$$\frac{\mathrm{d}\bar{M}_t}{\mathrm{d}\phi} = \frac{\rho I_t c^2 z^2}{EI_n}\bar{\Omega}_t + \frac{a}{c}\bar{M}_n + \bar{B}_{10}, \tag{24j}$$

$$\frac{\mathrm{d}\bar{M}_n}{\mathrm{d}\phi} = \frac{\rho c^2 z^2}{E}\bar{\Omega}_n + \bar{T}_b - \frac{a}{c}\bar{M}_t + \frac{h}{c}\bar{M}_b + \bar{B}_{11}, \tag{24k}$$

$$\frac{\mathrm{d}\bar{M}_b}{\mathrm{d}\phi} = \frac{\rho I_b c^2 z^2}{EI_n}\bar{\Omega}_b - \bar{T}_n - \frac{h}{c}\bar{M}_n + \bar{B}_{12}. \tag{24l}$$

Irrespective of the rod geometry, the following four cases may now be distinguished:
Case 1: Static loading, elastic material.
Case 2: Static loading, viscoelastic material (quasi-static case).
Case 3: Dynamic loading, elastic material.
Case 4: Dynamic loading, viscoelastic material.
In the cases of static loading, the terms including mass density in Eqs. (24g)–(24l) become null, irrespective of the rod material being elastic or viscoelastic. When the rod material is viscoelastic, elastic material constants are replaced with the viscoelastic counterparts as shown in Eqs. (17)–(18).

7. Solutions of the differential equations with the complementary functions method

Eqs. (24a)–(24l) make up a set of 12 simultaneous differential equations with constant coefficients. Each one of these equations involves first order derivatives with respect to position. The relationships given for the dynamic loading case in the Laplace space in Ref. [34] are modified to be used for the viscoelastic material cases. In matrix notation, Eqs. (24a)–(24l) can be expressed as

$$\frac{\mathrm{d}\bar{\mathbf{Y}}(\phi, z)}{\mathrm{d}\phi} = \bar{\mathbf{F}}(\phi, z)\bar{\mathbf{Y}}(\phi, z) + \bar{\mathbf{B}}(\phi, z). \tag{25}$$

For the case of a spatial bar, the elements of the state vector are defined as

$$\bar{\mathbf{Y}}(\phi, z) = \{\bar{\mathbf{U}}(\phi, z), \bar{\mathbf{\Omega}}(\phi, z), \bar{\mathbf{T}}(\phi, z), \bar{\mathbf{M}}(\phi, z)\}^{\mathrm{T}}. \tag{26}$$

The complementary functions method is based on the principle of solving Eq. (26) with the aid of initial conditions. This method is basically the reduction of two-point boundary value problems to the numerical solution of initial-value problems which are much more suitable for programming. The general solution of Eq. (26) is given by

$$\bar{\mathbf{Y}}(\phi, z) = \sum_{m=1}^{12} C_m(\bar{\mathbf{U}}^{(m)}(\phi, z)) + \bar{\mathbf{V}}(\phi, z), \tag{27}$$

where $\bar{\mathbf{U}}^{(m)}(\phi, z)$ is the complementary solution such that its mth component is equal to 1, with other components set equal to zero, $\bar{\mathbf{V}}(\phi, z)$ is the inhomogeneous solution with all zero initial conditions and the integration constants C_m will be determined from the boundary conditions at both ends.

8. Determination of the dynamic stiffness matrix

The element equation is given in the Laplace domain by

$$\{\bar{\mathbf{p}}\} = [\bar{\mathbf{k}}]\{\bar{\mathbf{d}}\} + \{\bar{\mathbf{f}}\}. \tag{28}$$

There are six degrees of freedom at each node; three translations and three rotations. Letting i stand for the beginning and j for the end of an element, the end displacements and the end forces are given as

$$\{\bar{\mathbf{d}}\}^{\mathrm{T}} = \{\bar{\mathbf{U}}(\phi_i, z), \bar{\mathbf{\Omega}}(\phi_i, z), \bar{\mathbf{U}}(\phi_j, z), \bar{\mathbf{\Omega}}(\phi_j, z)\}, \tag{29}$$

$$\{\bar{\mathbf{p}}\}^{\mathrm{T}} = \{\bar{\mathbf{T}}(\phi_i, z), \bar{\mathbf{M}}(\phi_i, z), \bar{\mathbf{T}}(\phi_j, z), \bar{\mathbf{M}}(\phi_j, z)\}. \tag{30}$$

In order to determine the element stiffness matrix, the end displacements of the element as defined in Eq. (29) are equated to unity for any one of the 12 directions while keeping the others zero. This is done 12 times using each equation. From the homogeneous solution of system (24), the element end forces are obtained, and these forces are incorporated into the element dynamic stiffness matrix.

The fixed-end forces are computed from Eq. (24) by taking all the end displacements to be equal to zero.

$$\{\bar{\mathbf{f}}\}^{\mathrm{T}} = \{-\bar{\mathbf{T}}(\phi_i, z), -\bar{\mathbf{M}}(\phi_i, z), \bar{\mathbf{T}}(\phi_j, z), \bar{\mathbf{M}}(\phi_j, z)\}. \tag{31}$$

For the transformation to the common reference system, the following equations are used:

$$[\bar{\mathbf{k}}]_{ijk} = [\mathbf{T}]^{\mathrm{T}}[\bar{\mathbf{k}}]_{tnb}[\mathbf{T}], \tag{32}$$

$$\{\bar{\mathbf{f}}\}_{ijk} = [\mathbf{T}]^{\mathrm{T}}\{\bar{\mathbf{f}}\}_{tnb}, \tag{33}$$

where the transformation matrix [**T**] is given by

$$[\mathbf{T}] = \begin{bmatrix} [\mathbf{B}(\phi_i)] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{B}(\phi_i)] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{B}(\phi_j)] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{B}(\phi_j)] \end{bmatrix}_{12\times 12}, \tag{34}$$

and [**B**] is defined in Eq. (22).

In this study, both the element dynamic stiffness matrix $\lfloor \bar{\mathbf{k}} \rfloor$ and the fixed-end forces $\{\bar{\mathbf{f}}\}$ are determined by solving Eq. (24) by the Complementary Functions Method in the Laplace domain. The system of the equations of motion can then be assembled from the element dynamic stiffness matrices and end forces as

$$[\mathbf{K(z)}]\{\mathbf{D}\} = \{\mathbf{P(z)}\}, \tag{35}$$

where $[\mathbf{K(z)}]$ and $\{\mathbf{P(z)}\}$ are the system dynamic stiffness matrix and the load vector. $\{\mathbf{D}\}$ is the vector of unknown displacements of the system.

9. Numerical example

In this study, a general-purpose computer program is coded in FORTRAN77 for time-dependent loadings in order to analyze quasi-static and dynamic responses of cylindrical helical rods made of linear viscoelastic materials. Butcher's fifth order Runge–Kutta algorithm [41] is used for the solution of the initial value problem based on the complementary function method. Forty steps of integration are used in the analysis. The Durbin's inverse Laplace transform [35,36] is applied for transformation from the Laplace domain to the time domain.

Example. A cantilever helical rod is now considered. The parameters used in this example are those used in Ref. [34] for the elastic material. Various dynamic loads are applied on the free end of the rod. Material and geometrical properties are: $d = 12$ cm, $\alpha = 25.522834°$, $a = 200$ cm, $E = 2.06 \times 10^{11}$ N/m^2, $\rho = 7850$ kg/m^3 and $\nu = 0.3$ (see Fig. 3). Various dynamic loads with amplitude $P_o = 10^6$ N are applied vertically at the free end of the rod. A time increment Δt of 20 ms is used in the calculations. An internal damping coefficient $g = 0.02$ is used for all cases. It is well documented that increasing the damping coefficient decreases the amplitude of the dynamic response [9,11].

Non-dimensional vertical displacement at the free end and non-dimensional shear force bending moment at the fixed end, are shown in Figs. 4a–d and 5a–d for different loading cases.

The dynamic behaviour of the viscoelastic helical bar will eventually disappear and it will approach the static state. The moment Mz is equal to zero under static loads. However, in the case of dynamic loads, due to inertia forces it assumes values different from zero (see Figs. 4d, 5d).

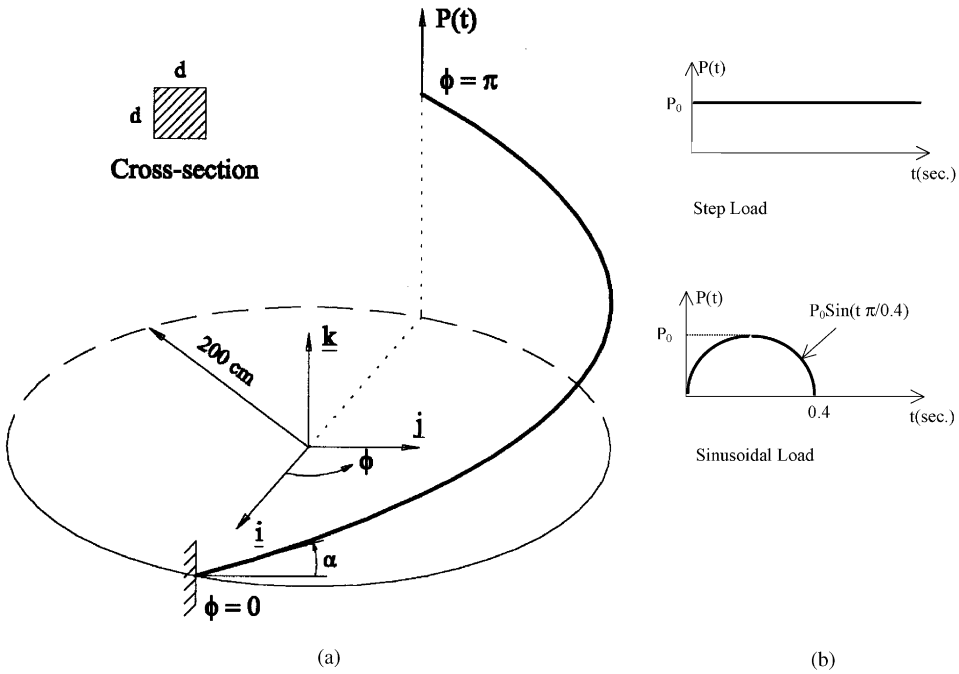

Fig. 3. (a) A cantilever helical rod; (b) type of dynamic loads.

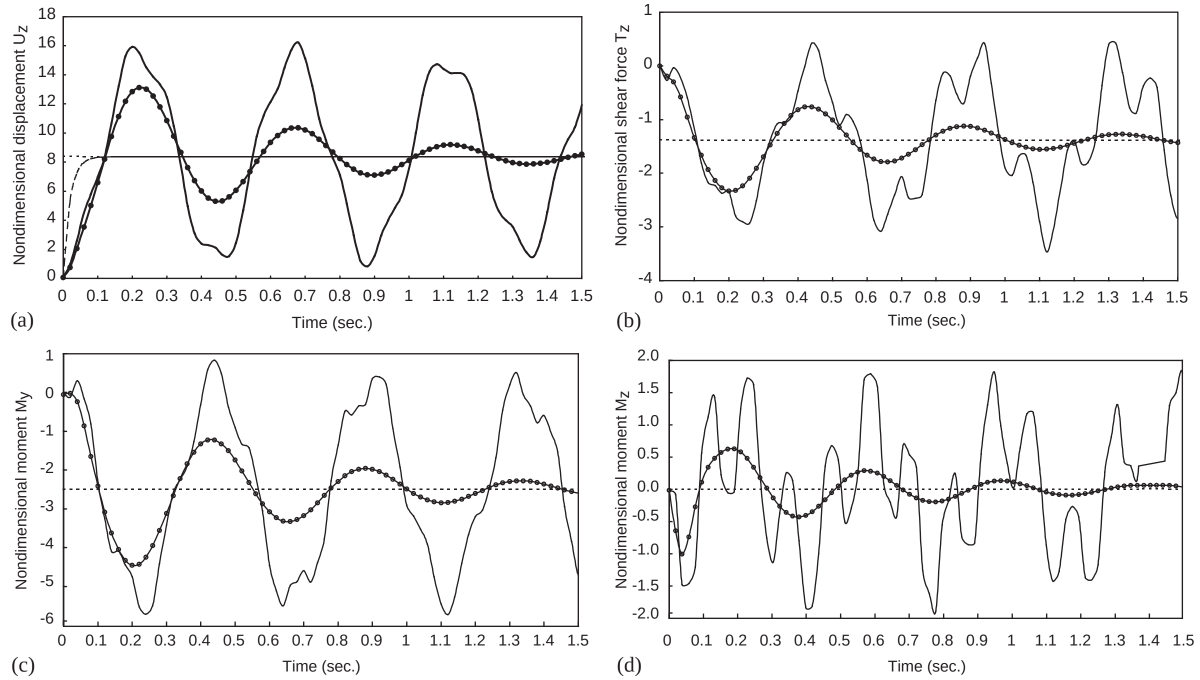

Fig. 4. (a) Vertical displacement versus time at the free end for step load. (b) Shear force versus time at the fixed end for step load. (c) Moment My versus time at the fixed end for step load. (d) Moment Mz versus time at the fixed end for step load. (······ Static; - - - Quasi-Static ($g = 0.02$); —— Elastic-Dynamic; —○— Viscoelastic ($g = 0.02$)).

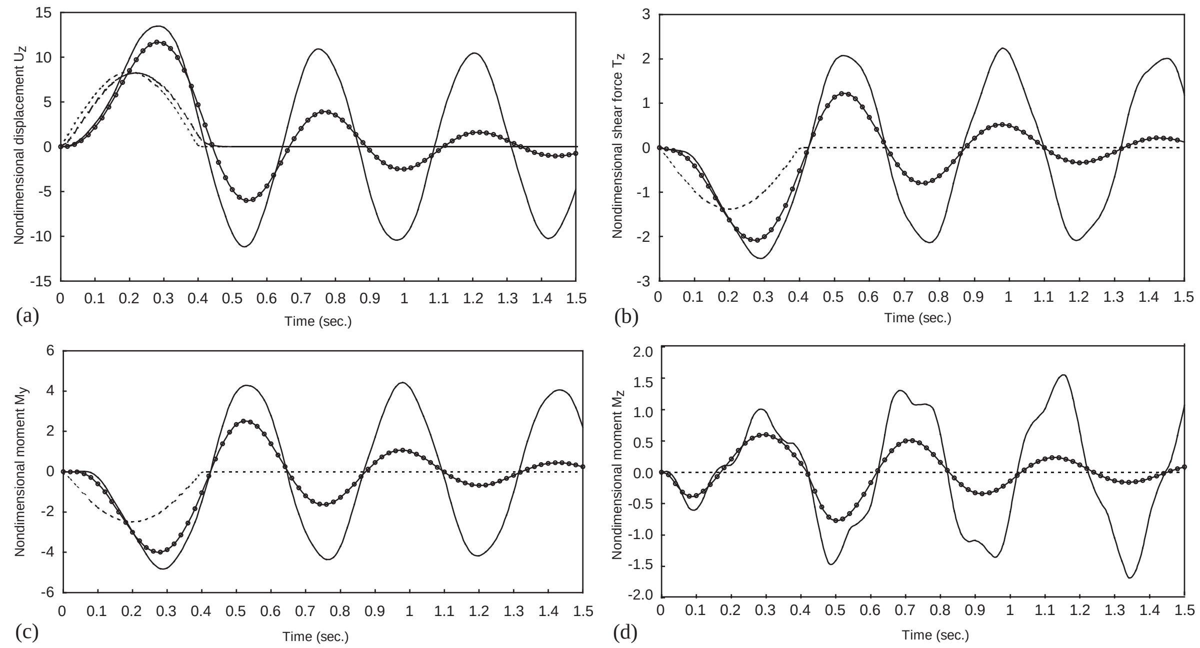

Fig. 5. (a) Vertical displacement versus time at the free end for sinusoidal impulsive load. (b) Shear force versus time at the fixed end for sinusoidal impulsive load. (c) Moment My versus time at the fixed end for sinusoidal impulsive load. (d) Moment Mz versus time at the fixed end for sinusoidal impulsive load. (······ Static; - - - Quasi-Static ($g = 0.02$); —— Elastic-Dynamic; —○— Viscoelastic ($g = 0.02$)).

10. Discussions and conclusions

The quasi-static and dynamic response of cylindrical helical rods made of linear viscoelastic materials are investigated in the Laplace domain in this study. In the viscoelastic material case, according to the correspondence principle, the material constants are replaced with their complex counterparts in the Laplace domain. Viscoelastic counterparts of both the Poisson ratio and elasticity modulus are used in the formulation.

In the present study, the dynamic stiffness matrix has been calculated in the Laplace domain by applying the complementary functions method to the differential equations in canonical form. This provides great convenience in the solution of the physical problems having general boundary conditions. Another advantage of using the complementary functions method-based solution is that the helical rods with variable cross-section and geometry, which yield ordinary differential equations having variable coefficients, can also be considered. The differential equations can be solved by using the complementary functions method with sufficient accuracy as required with an appropriate integration step-size.

The quasi-static and dynamic behaviour of cylindrical helical rods are investigated by using the Kelvin model for viscoelastic materials. The dynamic behaviour of the viscoelastic helical bar will disappear after some time, approaching the static state. The time to reach the static behaviour is proportional to the damping coefficient. The damping effects in viscoelastic material reduce the peak values of the dynamic response.

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References

[1] W. Flügge, *Viscoelasticity*, 2nd Edition, Springer, Berlin, Heidelberg, 1975.
[2] E. Kiral, T. Tokdemir, S. Ural, Transient response of an elastic or/and viscoelastic curved rod under arbitrary time dependent loading, *METU Journal of Pure and Applied Sciences* 9 (1976) 63–86.
[3] W.N. Findley, J.S. Lai, K. Onaran, *Creep and Relaxation of Nonlinear Viscoelastic Material*, North-Holland, New York, 1976, 109pp.
[4] R.M. Christensen, *Theory of Viscoelasticity*, 2nd Edition, Academic Press, New York, 1982, 24pp.
[5] J.L. White, Finite element in linear viscoelasticity, *Proceeding of the Second Conference on Matrix Method in Structural Mechanics*, AFFDL-TR-68-150, 1986, pp. 489–516.
[6] R.A. Adey, C.A. Brebbia, Efficient method for solution of viscoelastic problem, *Journal of Engineering Mechanics Division American Society of Civil Engineers* 99 (1973) 1119–1127.
[7] W.H. Chen, T.C. Lin, Dynamic analysis of viscoelastic structure using incremental finite element method, *Computers and Structures* 4 (1982) 271–276.
[8] Y. Yamada, H. Takabatake, T. Sato, Effect of time dependent material properties or dynamic response, *International Journal for Numerical Methods in Engineering* 8 (1974) 403–414.
[9] T. Chen, The hybrid Laplace transform/finite element method applied to the quasi-static and dynamic analysis of viscoelastic Timoshenko beams, *International Journal for Numerical Methods in Engineering* 38 (1995) 509–522.

[10] A.Y. Aköz, F. Kadioğlu, The mixed finite element solutions of circular beam on elastic foundation, *Computers and Structures* 60 (4) (1996) 643–651.
[11] A.Y. Aköz, F. Kadioğlu, The mixed finite element method for the quasi-static and dynamic analysis of viscoelastic Timoshenko beams, *International Journal for Numerical Methods in Engineering* 44 (1999) 1909–1932.
[12] F. Kadioğlu, A.Y. Aköz, The mixed finite element method for the dynamic analysis of viscoelastic circular beams, *Fourth International Conference on Vibration Problems*, Calcutta, India, November 27–30, 1999, pp. 40–52.
[13] M.H. Ilyasov, A.Y. Aköz, The vibration and dynamic stability of viscoelastic plates, *International Journal of Engineering Sciences* 38 (2000) 695–714.
[14] S.W. Park, R.A. Schapery, Methods of interconversion between linear viscoelastic material function. Part I-a numerical method based on Prony series, *International Journal of Solids and Structures* 36 (1999) 1653–1675.
[15] R.A. Schapery, S.W. Park, Methods of interconversion between linear viscoelastic material function. Part II—an approximate analytical method, *International Journal of Solids and Structures* 36 (1999) 1677–1699.
[16] S.W. Park, Analytical modeling of viscoelastic dampers for structural and vibration control, *International Journal of Solids and Structures* 38 (2001) 8065–8092.
[17] T.W. Kim, J.H. Kim, Parametric instability of a cross-ply laminated beam with viscoelastic properties under a periodic force, *Composite Structures* 51 (2001) 205–209.
[18] S. Banerjee, S.K. Roychoudhuri, Spherically symmetric thermo-visco-elastic waves in a visco-elastic medium with a spherical cavity, *Computers and Mathematics Applications* 36 (1) (1995) 91–98.
[19] L.B. Eldred, W.P. Baker, A.N. Palazotto, Numerical application of frictional derivative model constitutive relations for viscoelastic materials, *Computers and Structures* 60 (6) (1996) 875–882.
[20] N. Huber, C. Tsakmakis, Finite deformation viscoelasticity laws, *Mechanics of Materials* 32 (2000) 1–18.
[21] M. Enelund, L. Mähler, K. Runesson, B.L. Josefson, Formulation and integration of the standard linear viscoelastic solid with fractional order rate laws, *International Journal of Solids and Structures* 36 (1999) 2417–2442.
[22] R. Pavlović, P. Kozić, P. Rajković, Influence of transverse shear on stochastic instability of viscoelastic beam, *International Journal of Solids and Structures* 38 (2001) 6829–6837.
[23] N.H. Scott, Energy flux and dissipation of inhomogeneous plane wave forms in Kelvin-Voigt viscoelasticity, *International Journal Engineering Sciences* 35 (6) (1997) 561–572.
[24] F.A. Sturla, A. Argento, Free and forced vibrations of spinning viscoelastic beam, *Journal of Vibration and Acoustics—Transactions of the American Society of Mechanical Engineers* 118 (3) (1996) 463–468.
[25] H.G. Georgiadis, A.P. Rigatos, N.C. Charalambakis, Dynamic stress-concentration around a hole in a viscoelastic plate, *Acta Mechanica* 111 (1–2) (1995) 1–12.
[26] I. Carol, Z.P. Bazant, Viscoelasticity with aging caused by solidification of nonaging constituent, *Journal of Engineering Mechanics—American Society of Civil Engineers* 119 (11) (1993) 2252–2269.
[27] V. Yildirim, N. İnce, Natural frequencies of helical springs of arbitrary shape, *Journal of Sound and Vibration* 204 (2) (1997) 311–329.
[28] J. Lee, D.J. Thompson, Dynamic Stiffness formulation, free vibration and wave motion of helical springs, *Journal of Sound and Vibration* 239 (2) (2001) 297–320.
[29] V.L. Berdichevsky, V.G. Sutyrin, Problem of an equivalent rod in nonlinear theory of springs, *PMM* 47 (2) (1984) 197–205.
[30] V.L. Berdichevsky, L.A. Starosel'skii, On the theory of curvilinear Timoshenko type rods, *PMM* 47 (6) (1983) 809–817.
[31] C.E.S. Cesnik, D.H. Hodges, V.G. Sutyrin, Cross-sectional analysis of composite beams including large initial twist and curvature effects, *American Institute of Aeronautics and Astronautics Journal* 34 (9) (1996) 1913–1920.
[32] M. Borri, G.L. Ghiringhelli, T. Merlini, Linear analysis of naturally curved and twisted anisotropic beams, *Composites Engineering* 2/5 (7) (1992) 433–456.
[33] W. Yu, D.H. Hodges, V.V. Volovoi, C.E.S. Cesnik, On Timoshenko-like modeling of initially curved and twisted composite beams, *International Journal of Solids and Structures* 39 (19) (2002) 5101–5121.
[34] B. Temel, F.F. Calim, Forced vibration of cylindrical helical rods subjected to impulsive loads, *Journal of Applied Mechanics—American Society of Mechanical Engineers* 70 (2) (2003) 281–291.
[35] F. Durbin, Numerical inversion of Laplace transforms: an efficient improvement to Dubner and Abate's method, *Computer Journal* 17 (1974) 371–376.

[36] G.V. Narayanan, Numerical Operational Methods in Structural Dynamics, Ph.D. Thesis, University of Minnesota, Minneapolis, MN, 1979.
[37] H.R. Yerli, B. Temel, E. Kiral, Transient infinite elements for 2D soil-structure interaction analysis, *Journal of Geotechnical Geoenviromental Engineering* 124 (10) (1998) 976–988.
[38] I.S. Sokolnikoff, R.M. Redheffer, *Mathematics of Physics and Modern Engineering*, McGraw-Hill, Tokyo, 1958.
[39] E. Kiral, A. Ertepinar, Studies on elastic rods subject to diverse external agencies—Part III: vibrational analysis of space rods, *METU Journal of Pure and Applied Science* 7 (1) (1974) 55–69.
[40] B.A. Boley, J.H. Weiner, *Theory of Thermal Stresses*, Wiley, New York, 1960.
[41] S.C. Chapra, R.P. Canale, *Numerical Methods for Engineers*, 3rd Edition, McGraw-Hill, New York, 1998.